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NOTE

On Existence Theorems for Finite-Codimensional Subspaces in $C(Q)$[1]

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Some variants of the theorem on the existence of best approximation elements in a finite-codimensional subspace of the complex space $C(Q)$ of continuous functions are given. © 2002 Elsevier Science (USA)

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In approximation theory, existence subspaces of finite codimension have been studied since the 1960s. The following theorem was established by Garkavi [3] in the real case and by Vlasov [4] for the complex $C(Q)$. See also [5, 6].

THEOREM A. *In order that a finite-codimensional subspace $L \subset C(Q)$ be an existence set, it is necessary and sufficient that the following conditions be satisfied*:

(a) $\forall \mu \in L^{\perp} \setminus \{0\}$ *there exists a continuous Radon–Nikodým derivative* $\frac{d\mu}{d|\mu|}$;
(b) $\forall \mu, \nu \in L^{\perp} \setminus \{0\}$ *the set* $S_\mu \setminus S_\nu$ *is closed*;
(c) $\forall \mu, \nu \in L^{\perp} \setminus \{0\}$ *there exists a usual Radon–Nikodým derivative* $\frac{d\mu}{d\nu} \in L_1(S_\nu, |\nu|)$.

The purpose of this note is to formulate the existence theorem in its different variants.

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We denote by S_μ the support of $\mathbb{K}$-valued Radon measure $\mu \in C(Q, \mathbb{K})^*$. Here and in the sequel, $\mathbb{K}$ denotes both the real ($\mathbb{R}$) and complex ($\mathbb{C}$) field. By definition, $f = d\mu/dv$ if $\mu e = \int_e f\, dv$ for any Borel set e. The following properties of a Radon–Nikodým derivative should be noted (see, for example, [2, Chap. III, Subsection 10]):

LEMMA 1. *If $f = d\mu/d|\mu|$, then $|f(t)| = 1$ μ-almost everywhere; if f is continuous on S_μ, then $|f(t)| = 1$ for all $t \in S_\mu$. In addition,*

$$\frac{d\mu}{d|v|} = \frac{d\mu}{dv} \cdot \frac{dv}{d|v|}$$

μ-almost everywhere on the set S_v.

The symbol $\tilde{\mathbb{K}}$ denotes $\mathbb{K}$ supplemented by "ideal elements": $\tilde{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty w : |w| = 1\}$; $C^\infty(Q, \tilde{\mathbb{K}})$ denotes the set of functions from Q to $\tilde{\mathbb{K}}$ continuous with respect to the corresponding topologies in Q and $\tilde{\mathbb{K}}$ (see [5]: it is assumed that $z_n \to \infty w$ iff $|z_n| \to \infty$ and $z_n/|z_n| \to w$; $\infty w_n \to \infty w$ as $w_n \to w$, $(\infty w) \cdot 0 = 0$, $\infty \cdot 0 = 0$). Topologically, $\tilde{\mathbb{C}}$ is equivalent to the disk $|z| \leqslant 1$.

The set $C^\infty(Q, \tilde{\mathbb{K}})$ is neither linear nor normed space, and unfortunately our previous notation $C(Q, \tilde{\mathbb{C}})$ [5] can cause misunderstanding. Existence subspaces are often called *proximinal.*

The following lemmas from [5] will be needed in the sequel.

LEMMA 2 (Vlasov [5, Lemmas 5 and 6]). *Let conditions* (a) *and* (c) *be fulfilled. Then for any $\mu, v \in L^\perp \setminus \{0\}$ there exists a Radon–Nikodým derivative $d\mu/dv \in C^\infty(S_v, \tilde{\mathbb{K}})$.*

LEMMA 3 (Vlasov [5, Lemma 8]). *Let $\mu, v \in C(Q)^* \setminus \{0\}$, $z \in \mathbb{K}$, $\lambda = \mu - zv$, $f = d\mu/dv$, $g = dv/d|v|$. Then*

$$\frac{d\lambda}{d|\lambda|}(t) = \frac{f(t) - z}{|f(t) - z|} g(t)$$

for λ-almost all $t \in S_v$ with $f(t) \neq z$.

LEMMA 4 (Vlasov [5, Lemma 2]). *Let $\mu, v \in C(Q)^*$. Then there exists a countable set $R_{\mu v} \subset \mathbb{K}$ such that for all $z \in \mathbb{K} \setminus R_{\mu v}$, the measures μ and v are absolutely continuous with respect to the measure $\mu + zv$ and $S_{\mu + zv} = S_\mu \cup S_v$.*

THEOREM 1. *In order that a finite-codimensional subspace $L \subset C(Q)$ be an existence set it is necessary and sufficient that there hold the conditions* (b) and

(1) $\forall \mu, \nu \in L^{\perp} \setminus \{0\}$, *there exists a Radon–Nikodým derivative*

$$\frac{d\mu}{d|\nu|} \in C^{\infty}(S_\nu, \tilde{\mathbb{K}}).$$

Proof. *Necessity*: By Lemma 2, $d\mu/d\nu \in C^{\infty}(S_\nu, \tilde{\mathbb{K}})$, and by Lemma 1 condition (1) follows.

Sufficiency is valid since (1) implies (c) and in addition (a) by letting $\nu = \mu$. ∎

THEOREM 2. *The following condition is sufficient for the proximinality of a finite-codimensional subspace $L \subset C(Q, \mathbb{K})$:*

(2) $\forall \mu, \nu \in L^{\perp} \setminus \{0\}, \qquad \dfrac{d\mu}{d|\nu|} \in C(S_\nu, \mathbb{K}).$

Proof. Let us apply Theorem A. Condition (c) is fulfilled since $f = d\mu/d\nu = (d\mu/d|\nu|)/(d\nu d/|\nu|) \in C(S_\nu, \mathbb{K})$ by Lemma 1. Condition (2) implies (a) by letting $\nu = \mu$. To derive condition (b), we must show that $\overline{S_\mu \setminus S_\nu} \subset S_\mu \setminus S_\nu$, i.e. that $\overline{S_\mu \setminus S_\nu} \cap S_\nu = \varnothing$. Suppose the contrary: there exists $t \in \overline{S_\mu \setminus S_\nu} \cap S_\nu$. Then there exists a net $t_\alpha \in S_\mu \setminus S_\nu$, $t_\alpha \to t$. Consider a measure $\lambda = \mu - z\nu$, where $z \in \mathbb{K} \setminus \{0\}$ can be chosen in such a way that $z \neq f(t) = d\mu/d\nu(t), S_\lambda = S_\mu \cup S_\nu$ (see Lemma 4). Note that $f = d\mu/d\nu \in C(S_\nu, \mathbb{K})$ by Lemma 1. We have $\lambda = \mu$ on $S_\mu \setminus S_\nu$, and therefore

$$\frac{d\lambda}{d|\lambda|}(t_\alpha) = \frac{d\mu}{d|\mu|}(t_\alpha) \to \frac{d\mu}{d|\mu|}(t);$$

on the other hand, by Lemma 3 and the continuity of $d\lambda/d|\lambda|$ at the point t,

$$\frac{d\lambda}{d|\lambda|}(t) = \frac{f(t) - z}{|f(t) - z|} g(t) = \frac{d\mu}{d|\mu|}(t)$$

for any $z \notin R_{\mu\nu}, z \neq f(t)$, which is impossible: the fraction cannot be the same for various $z \notin R_{\mu\nu} \cup \{f(t)\}$ since $f(t) \in \mathbb{K}$. ∎

THEOREM 3. *In order that a finite-codimensional subspace $L \subset C(Q, \mathbb{K})$ be an existence set it is sufficient that there hold conditions* (a) *and*

(3) $\forall \mu, \nu \in L^{\perp} \setminus \{0\}$ *on the set* S_ν *there exists a bounded Radon–Nikodým derivative* $d\mu/d\nu \in L_\infty(S_\nu, |\nu|)$.

Proof. Since (3) implies (c), by Lemma 2 there exists a Radon–Nikodým derivative $d\mu/d\nu \in C^\infty(S_\nu, \tilde{\mathbb{K}})$. By (3) $d\mu/d\nu \in C(S_\nu, \mathbb{K})$. Taking into account Lemma 1, we obtain (2) and hence the result by Theorem 2. ∎

Remark. In the real case, condition (2) is equivalent to conditions (1) and (2) in [1]. Unfortunately, these authors erroneously believe, that the Radon–Nikodým derivatives are necessarily bounded. That this is not the case is shown by a simple example in $C(Q, \mathbb{R})$, where $Q = [0,1]$, $Q_1 = \{t_i\}_{i=1}^{\infty}$, $t_i = 1/i, \mu\{t_i\} = 1/i^2$, $\nu\{t_i\} = 1/i^3$, $f(t_i) = d\mu/d\nu(t_i) = i = 1, 2, \ldots \to \infty$; $|\mu|(Q \setminus Q_1) = |\nu|(Q \setminus Q_1) = 0$; the subspace $L = \{\mu, \nu\}_\perp$ is proximinal by Garkavi's theorem. It is not difficult to change this example in such a way that the measures μ and ν be atomless.

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